arXiv:0705.0834v2 [math.RT] 8 Nov 2008

UNIVERSAL DEFORMATION RINGS AND DIHEDRAL 2-GROUPS

FRAUKE M. BLEHER

Abstract. Let k be an algebraically closed field of characteristic 2, and let W be the ring of infinite Witt vectors over k. Suppose D is a dihedral 2-group. We prove that the universal deformation ring $R(D,V)$ of an endo-trivial kD-module V is always isomorphic to $W[\mathbb{Z}/2\times\mathbb{Z}/2]$. As a consequence we obtain a similar result for modules V with stable endomorphism ring k belonging to an arbitrary nilpotent block with defect group D. This confirms for such V conjectures on the ring structure of the universal deformation ring of V which had previously been shown for V belonging to cyclic blocks or to blocks with Klein four defect groups.

1. Introduction

Let k be an algebraically closed field of positive characteristic p, let $W = W(k)$ be the ring of infinite Witt vectors over k, and let G be a finite group. There are various classical results in the literature concerning the lifting of finitely generated kG-modules over complete local commutative Noetherian rings with residue field k, such as Green's liftability theorem. To understand and generalize these results, it is useful to reformulate them in terms of deformation rings. For example, Alperin has proved in [2] that in case G is a p-group, every endo-trivial kG-module V can be lifted to an endo-trivial WG-module. This can be reformulated as saying that for every such V, there is a surjection from the universal deformation ring $R(G,V)$ to W. A natural question is then to determine $R(G,V)$ itself. In this paper, we determine the universal deformation rings $R(D,V)$ for all endo-trivial kD-modules V when k has characteristic $p=2$ and D is a dihedral 2-group.

For arbitrary p and G, a finitely generated kG-module V is called endo-trivial if the kG-module $\mathrm{Hom}_k(V,V)\cong V^*\otimes_k V$ is isomorphic to a direct sum of the trivial simple kG-module k and a projective kG-module. Endo-trivial modules play an important role in the modular representation theory of finite groups, in particular in the context of derived equivalences and stable equivalences of block algebras, and also as building blocks for the more general endo-permutation modules, which for many groups are the sources of the simple modules (see e.g. [17, 27]). In [15, 14], Carlson and Thévenaz classified all endo-trivial kG-modules when G is a p-group. Since by [12], the endo-trivial kG-modules V of a p-group G are precisely the modules whose

1991 *Mathematics Subject Classification.* Primary 20C20; Secondary 20C15, 16G10.

Key words and phrases. Universal deformation rings, dihedral groups, endo-trivial modules, nilpotent blocks.

The author was supported in part by NSA Grant H98230-06-1-0021 and NSF Grant DMS06-51332.

stable endomorphism ring is one-dimensional over k, it follows by [6, Prop. 2.1] that V has a well-defined universal deformation ring $R(G, V)$.

The topological ring $R(G, V)$ is universal with respect to deformations of V over complete local commutative Noetherian rings R with residue field k. A deformation of V over such a ring R is given by the isomorphism class of a finitely generated RG-module M which is free over R, together with a kG-module isomorphism $k \otimes_R M \to V$ (see §2). Note that all these rings R, including $R(G, V)$, have a natural structure as W-algebras.

In number theory, the main motivation for studying universal deformation rings for finite groups is to provide evidence for and counter-examples to various possible conjectures concerning ring theoretic properties of universal deformation rings for profinite Galois groups. The idea is that universal deformation rings for finite groups can be more easily described using deep results from modular representation theory due to Brauer, Erdmann [19], Linckelmann [21, 22], Butler-Ringel [10] and others. Moreover, the results in [18] show that if Γ is a profinite group and V is a finite dimensional k-vector space with a continuous Γ-action which has a universal deformation ring, then $R(\Gamma, V)$ is the inverse limit of the universal deformation rings $R(G, V)$ when G runs over all finite discrete quotients of Γ through which the Γ-action on V factors. Thus to answer questions about the ring structure of $R(\Gamma, V)$, it is natural to first consider the case when $\Gamma = G$ is finite. Later in the introduction we will discuss some number theoretic problems which originate from considering how our results for finite groups arise from arithmetic.

Suppose now that G is an arbitrary finite group and V is a kG-module such that the stable endomorphism ring $\underline{\mathrm{End}}_{kG}(V)$ is one-dimensional over k, i.e. V has a well-defined universal deformation ring $R(G, V)$. The results in [6] led to the following question relating the universal deformation rings $R(G, V)$ to the local structure of G given by defect groups of blocks of kG.

Question 1.1. *Let B be a block of kG with defect group D, and suppose V is a finitely generated kG-module with stable endomorphism ring k such that the unique (up to isomorphism) non-projective indecomposable summand of V belongs to B. Is the universal deformation ring $R(G, V)$ of V isomorphic to a subquotient ring of the group ring WD?*

The results in [6, 4, 5] show that this question has a positive answer in case B is a block with cyclic defect groups, i.e. a block of finite representation type, or a tame block with Klein four defect groups, or a tame block with dihedral defect groups which is Morita equivalent to the principal 2-modular block of a finite simple group. For the latter type of blocks, there are precisely three isomorphism classes of simple modules.

In [7, 8], it was shown that if $p = 2$, G is the symmetric group S_4 and E is a 2-dimensional simple kS_4-module then $R(G, E) \cong W[t]/(t^2, 2t)$, giving an example of a universal deformation ring which is not a complete intersection, thus answering a question of M. Flach [16]. A new proof of this result has been given in [11] using only elementary obstruction calculus. In [7, 8], it was additionally shown that this example arises from arithmetic in the following way. There are infinitely many real quadratic

fields L such that the Galois group $G_{L,\emptyset}$ of the maximal totally unramified extension of L surjects onto S_4 and $R(G_{L,\emptyset}, E) \cong R(S_4, E) \cong W[t]/(t^2, 2t)$ is not a complete intersection, where E is viewed as a module of $G_{L,\emptyset}$ via inflation. The universal deformation rings in [6, 4] are all complete intersections, whereas the results in [5] provide an infinite series of G and V for which $R(G, V)$ is not a complete intersection.

In this paper, we consider the endo-trivial modules for the group ring kD when k has characteristic 2 and D is a dihedral 2-group of order at least 8. Note that kD is its own block and the trivial simple module k is the unique simple kD-module up to isomorphism. Our main result is as follows, where Ω denotes the syzygy, or Heller, operator (see for example [1, §20]).

Theorem 1.2. *Let k be an algebraically closed field of characteristic 2, let $d \geq 3$, and let D be a dihedral group of order 2^d. Suppose V is a finitely generated endo-trivial kD-module.*

i. *If V is indecomposable and $\mathfrak{C}$ is the component of the stable Auslander-Reiten quiver of kD containing V, then $\mathfrak{C}$ contains either k or $\Omega(k)$, and all modules belonging to $\mathfrak{C}$ are endo-trivial.*
ii. *The universal deformation ring $R(D, V)$ is isomorphic to $W[\mathbb{Z}/2 \times \mathbb{Z}/2]$. Moreover, every universal lift U of V over $R = R(D, V)$ is endo-trivial in the sense that the RD-module $U^* \otimes_R U$ is isomorphic to the direct sum of the trivial RD-module R and a free RD-module.*

In particular, $R(D, V)$ is always a complete intersection and isomorphic to a quotient ring of the group ring WD.

It is a natural question to ask whether Theorem 1.2 can be used to construct deformation rings arising from arithmetic. More precisely, let L be a number field, let S be a finite set of places of L, and let L_S be the maximal algebraic extension of L unramified outside S. Denote by $G_{L,S}$ the Galois group of L_S over L. Suppose k has characteristic p, G is a finite group and V is a finitely generated kG-module with stable endomorphism ring k. As in [8], one can ask whether there are L and S such that there is a surjection $\psi : G_{L,S} \to G$ which induces an isomorphism of deformation rings $R(G_{L,S}, V) \to R(G, V)$ when V is viewed as a representation for $G_{L,S}$ via ψ. It was shown in [8] that a sufficient condition for $R(G_{L,S}, V) \to R(G, V)$ to be an isomorphism for all such V is that $\mathrm{Ker}(\psi)$ has no non-trivial pro-p quotient. If this condition is satisfied, we say the group G caps L for p at S.

As mentioned earlier, this arithmetic problem was considered in [8] for the symmetric group S_4 in case $p = 2$. In fact, it was shown that there are infinitely many real quadratic fields L such that S_4 caps L for $p = 2$ at $S = \emptyset$. Since the Sylow 2-subgroups of S_4 are isomorphic to a dihedral group D_8 of order 8, this can be used to show that D_8 caps infinitely many sextic fields L' for $p = 2$ at $S = \emptyset$. In particular, $R(G_{L',\emptyset}, V) \cong R(D_8, V) \cong W[\mathbb{Z}/2 \times \mathbb{Z}/2]$ for all endo-trivial kD_8-modules V. Since the fields L' have degree 6 over $\mathbb{Q}$, this raises the question of whether one can replace L' by smaller degree extensions. Another question is if one can find similar results for dihedral groups D of arbitrary 2-power order. As in the proof of [8, Thm. 3.7(i)], one can show that D does not cap $\mathbb{Q}$ for $p = 2$ at any finite set S of rational

primes. Hence the best possible results one can expect to be valid for all endo-trivial kD-modules should involve extensions L of $\mathbb{Q}$ of degree at least 2.

We now discuss the proof of Theorem 1.2. As stated earlier, the endo-trivial kD-modules are precisely the kD-modules whose stable endomorphism ring is one-dimensional over k. The results of [13, §5] show that the group $T(D)$ of equivalence classes of endo-trivial kD-modules is generated by the classes of the relative syzygies of the trivial simple kD-module k. To prove part (i) of Theorem 1.2 we relate this description for indecomposable endo-trivial kD-modules to their location in the stable Auslander-Reiten quiver of kD. For part (ii) of Theorem 1.2, suppose $D = \langle \sigma, \tau \rangle$ where σ and τ are two elements of order 2 and $\sigma\tau$ has order 2^{d-1}, and let V be an indecomposable endo-trivial kD-module. We prove that there exists a continuous local W-algebra homomorphism $\alpha : W[\mathbb{Z}/2 \times \mathbb{Z}/2] \to R(D, V)$ by considering restrictions of V to $\langle \sigma \rangle$ and $\langle \tau \rangle$. We then analyze the kD-module structures of all lifts of V over the dual numbers $k[\epsilon]/(\epsilon^2)$ to show that α is in fact surjective. Using the ordinary irreducible representations of D, we prove that there are four distinct continuous W-algebra homomorphisms $R(D, V) \to W$ and show that this implies that α is an isomorphism.

In [9, 25], Broué and Puig introduced and studied so-called nilpotent blocks. Using [25], we obtain the following result as an easy consequence of Theorem 1.2.

Corollary 1.3. *Let k and D be as in Theorem 1.2. Let G be a finite group, and let B be a nilpotent block of kG with defect group D. Suppose V is a finitely generated B-module with stable endomorphism ring k. Then the universal deformation ring $R(G, V)$ is isomorphic to $W[\mathbb{Z}/2 \times \mathbb{Z}/2]$.*

The paper is organized as follows: In §2, we provide some background on universal deformation rings for finite groups. In §3, we study some subquotient modules of the free kD-module of rank 1 and describe lifts of two such kD-modules over W using the ordinary irreducible representations of D. In §4, we describe the locations of the indecomposable endo-trivial kD-modules in the stable Auslander-Reiten quiver of kD using [3, 13]. In §5, we complete the proof of Theorem 1.2 and Corollary 1.3.

The author would like to thank K. Erdmann and M. Linckelmann for helpful discussions on nilpotent blocks. She would also like to thank the referee for very useful comments which simplified some of the proofs.

2. Preliminaries

Let k be an algebraically closed field of characteristic $p > 0$, let W be the ring of infinite Witt vectors over k and let F be the fraction field of W. Let $\mathcal{C}$ be the category of all complete local commutative Noetherian rings with residue field k. The morphisms in $\mathcal{C}$ are continuous W-algebra homomorphisms which induce the identity map on k.

Suppose G is a finite group and V is a finitely generated kG-module. If R is an object in $\mathcal{C}$, a finitely generated RG-module M is called a lift of V over R if M is free over R and $k \otimes_R M \cong V$ as kG-modules. Two lifts M and M' of V over R are said to be isomorphic if there is an RG-module isomorphism $\alpha : M \to M'$ which respects

the kG-module isomorphisms $k \otimes_R M \cong V$ and $k \otimes_R M' \cong V$. The isomorphism class of a lift of V over R is called a deformation of V over R, and the set of such deformations is denoted by $\mathrm{Def}_G(V,R)$. The deformation functor $F_V : \mathcal{C} \to \mathrm{Sets}$ is defined to be the covariant functor which sends an object R in $\mathcal{C}$ to $\mathrm{Def}_G(V,R)$.

In case there exists an object $R(G,V)$ in $\mathcal{C}$ and a lift $U(G,V)$ of V over $R(G,V)$ such that for each R in $\mathcal{C}$ and for each lift M of V over R there is a unique morphism $\alpha : R(G,V) \to R$ in $\mathcal{C}$ such that $M \cong R \otimes_{R(G,V),\alpha} U(G,V)$, then $R(G,V)$ is called the universal deformation ring of V and the isomorphism class of the lift $U(G,V)$ is called the universal deformation of V. In other words, $R(G,V)$ represents the functor F_V in the sense that F_V is naturally isomorphic to $\mathrm{Hom}_{\mathcal{C}}(R(G,V),-)$. If $R(G,V)$ and the universal deformation corresponding to $U(G,V)$ exist, then they are unique up to unique isomorphism. For more information on deformation rings see [18] and [24].

The following four results were proved in [6] and in [5], respectively. Here Ω denotes the syzygy, or Heller, operator for kG (see for example [1, §20]).

Proposition 2.1. [6, Prop. 2.1] *Suppose V is a finitely generated kG-module with stable endomorphism ring $\underline{\mathrm{End}}_{kG}(V) = k$. Then V has a universal deformation ring $R(G,V)$.*

Lemma 2.2. [6, Cors. 2.5 and 2.8] *Let V be a finitely generated kG-module with stable endomorphism ring $\underline{\mathrm{End}}_{kG}(V) = k$.*

i. *Then $\underline{\mathrm{End}}_{kG}(\Omega(V)) = k$, and $R(G,V)$ and $R(G,\Omega(V))$ are isomorphic.*
ii. *There is a non-projective indecomposable kG-module V_0 (unique up to isomorphism) such that $\underline{\mathrm{End}}_{kG}(V_0) = k$, V is isomorphic to $V_0 \oplus P$ for some projective kG-module P, and $R(G,V)$ and $R(G,V_0)$ are isomorphic.*

Lemma 2.3. [5, Lemma 2.3.2] *Let V be a finitely generated kG-module such that there is a non-split short exact sequence of kG-modules*

$$0 \to Y_2 \to V \to Y_1 \to 0$$

with $\mathrm{Ext}^1_{kG}(Y_1,Y_2) = k$. Suppose that there exists a WG-module X_i which is a lift of Y_i over W for $i = 1,2$. Suppose further that

$$\dim_F \mathrm{Hom}_{FG}(F \otimes_W X_1, F \otimes_W X_2) = \dim_k \mathrm{Hom}_{kG}(Y_1,Y_2) - 1.$$

Then there exists a WG-module X which is a lift of V over W.

3. The dihedral 2-groups D

Let $d \geq 3$ and let D be a dihedral group of order 2^d given as

$$D = \langle \sigma, \tau \mid \sigma^2 = 1 = \tau^2, (\sigma\tau)^{2^{d-2}} = (\tau\sigma)^{2^{d-2}} \rangle.$$

Let k be an algebraically closed field of characteristic $p = 2$. The trivial simple kD-module k is the unique irreducible kD-module up to isomorphism. The free kD-module kD of rank one is indecomposable and its radical series has length $2^{d-1}+1$. The radical of kD is generated as a kD-module by $(1+\sigma)$ and by $(1+\tau)$, and

the socle of kD is one-dimensional over k and generated by $[(1+\sigma)(1+\tau)]^{2^{d-2}} = [(1+\tau)(1+\sigma)]^{2^{d-2}}$. Hence

$$\begin{aligned} \mathrm{rad}(kD) &= kD(1+\sigma) + kD(1+\tau), \\ \mathrm{soc}(kD) &= kD(1+\sigma) \cap kD(1+\tau) = k\,[(1+\sigma)(1+\tau)]^{2^{d-2}}. \end{aligned} \tag{3.1}$$

From this description it follows that $\mathrm{rad}(kD)/\mathrm{soc}(kD)$ is isomorphic to the direct sum of two indecomposable kD-modules, namely

$$\mathrm{rad}(kD)/\mathrm{soc}(kD) \cong kD(1+\sigma)/\mathrm{soc}(kD) \,\oplus\, kD(1+\tau)/\mathrm{soc}(kD). \tag{3.2}$$

Moreover, we have the following isomorphisms of kD-modules:

$$\begin{aligned} kD(1+\sigma) &\cong kD \otimes_{k\langle\sigma\rangle} k = \mathrm{Ind}_{\langle\sigma\rangle}^{D}\, k, \\ kD(1+\tau) &\cong kD \otimes_{k\langle\tau\rangle} k = \mathrm{Ind}_{\langle\tau\rangle}^{D}\, k. \end{aligned} \tag{3.3}$$

Let $\nu \in \{\sigma,\tau\}$, and define $E_\nu = kD(1+\nu)/\mathrm{soc}(kD)$. We have a commutative diagram of kD-modules of the form

$$\begin{tikzcd} 0 \arrow[r] & \Omega(E_\nu) \arrow[r, "\iota_\nu"] \arrow[d, "\Omega(f_\nu)"'] & kD \arrow[r, "\pi_\nu"] \arrow[d, "g_\nu"] & E_\nu \arrow[r] \arrow[d, "f_\nu"] & 0 \\ 0 \arrow[r] & \mathrm{soc}(kD) \arrow[r, "\iota"] & kD \arrow[r, "\pi"] & kD/\mathrm{soc}(kD) \arrow[r] & 0 \end{tikzcd} \tag{3.4}$$

where $\pi_\nu(1) = (1+\nu) + \mathrm{soc}(kD)$, $\pi(1) = 1 + \mathrm{soc}(kD)$, ι_ν and ι are inclusions, $g_\nu(1) = (1+\nu)$ and f_ν is induced by the inclusion map $kD(1+\nu) \hookrightarrow kD$. Since f_ν is injective, it follows that

$$\mathrm{Ker}(\Omega(f_\nu)) \cong \mathrm{Ker}(g_\nu) = kD(1+\nu). \tag{3.5}$$

We now turn to representations of D in characteristic 0. Let W be the ring of infinite Witt vectors over k, and let F be the fraction field of W. Let ζ be a fixed primitive 2^{d-1}-th root of unity in an algebraic closure of F. Then D has $4+(2^{d-2}-1)$ ordinary irreducible characters $\chi_1,\chi_2,\chi_3,\chi_4,\chi_{5,i}$, $1 \le i \le 2^{d-2}-1$, whose representations $\psi_1,\psi_2,\psi_3,\psi_4,\psi_{5,i}$, $1 \le i \le 2^{d-2}-1$, are described in Table 1. In fact, the

Table 1. The ordinary irreducible representations of D.

	σ	τ
$\psi_1 = \chi_1$	1	1
$\psi_2 = \chi_2$	-1	-1
$\psi_3 = \chi_3$	1	-1
$\psi_4 = \chi_4$	-1	1
$\psi_{5,i}$ $(1 \le i \le 2^{d-2}-1)$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \zeta^{-i} \\ \zeta^{i} & 0 \end{pmatrix}$

splitting field of D is $F(\zeta+\zeta^{-1})$, and the action of the Galois group $\mathrm{Gal}(F(\zeta+\zeta^{-1})/F)$ on the ordinary irreducible characters divides the characters $\chi_{5,i}$, $i = 1,\ldots,2^{d-2}-1$,

into $d-2$ Galois orbits $\mathcal{O}_0,\ldots,\mathcal{O}_{d-3}$ with $\mathcal{O}_\ell = \{\chi_{5,2^{d-3-\ell}(2u-1)} \mid 1 \le u \le 2^\ell\}$ for $0 \le \ell \le d-3$. Since the Schur index over F of each of the characters in these orbits is 1, we obtain $d-2$ non-isomorphic simple FD-modules $V_0,\ldots,V_{d-3}$ whose characters $\rho_0,\ldots,\rho_{d-3}$ satisfy

$$\rho_\ell = \sum_{u=1}^{2^\ell} \chi_{5,2^{d-3-\ell}(2u-1)} \qquad \text{for } 0 \le \ell \le d-3. \tag{3.6}$$

Moreover,

$$\mathrm{End}_{FD}(V_\ell) \cong F\left(\zeta^{2^{d-3-\ell}} + \zeta^{-2^{d-3-\ell}}\right) \qquad \text{for } 0 \le \ell \le d-3. \tag{3.7}$$

Lemma 3.1. *Let*

i. $(a,b) \in \{(1,3),(2,4)\}$, *or*
ii. $(a,b) \in \{(1,4),(2,3)\}$.

Let $N_{a,b}$ be an FD-module with character $\chi_a + \chi_b + \sum_{\ell=0}^{d-3} \rho_\ell$. Then there is a full WD-lattice $L_{a,b}$ in $N_{a,b}$ such that $L_{a,b}/2L_{a,b}$ is isomorphic to the kD-module $\mathrm{Ind}_{\langle\sigma\rangle}^D k$ in case (i) and to the kD-module $\mathrm{Ind}_{\langle\tau\rangle}^D k$ in case (ii).

Proof. Let X_σ be the set of left cosets of $\langle\sigma\rangle$ in D, and let WX_σ be the permutation module of D over W corresponding to X_σ. This means that WX_σ is a free W-module with basis $\{m_x \mid x \in X_\sigma\}$ and $g \in D$ acts as $g \cdot m_x = m_{gx}$ for all $x \in X_\sigma$. Using the formula for the permutation character associated to the FD-module FX_σ, we see that this character is equal to $\chi_1 + \chi_3 + \sum_{\ell=0}^{d-3} \rho_\ell$.

There is a surjective WD-module homomorphism $h_\sigma : WD \to WX_\sigma$ which is defined by $h_\sigma(1) = m_{\langle\sigma\rangle}$. Then $\mathrm{Ker}(h_\sigma) = WD(1-\sigma)$ and we have a short exact sequence of WD-modules which are free over W

$$0 \to WD(1-\sigma) \to WD \xrightarrow{h_\sigma} WX_\sigma \to 0. \tag{3.8}$$

Because the character of FD is $\chi_1 + \chi_2 + \chi_3 + \chi_4 + 2\sum_{\ell=0}^{d-3} \rho_\ell$, the character of $FD(1-\sigma)$ must be $\chi_2 + \chi_4 + \sum_{\ell=0}^{d-3} \rho_\ell$. Tensoring (3.8) with k over W, we obtain a short exact sequence of kD-modules

$$0 \to kD(1-\sigma) \to kD \to kX_\sigma \to 0. \tag{3.9}$$

Since $kX_\sigma \cong \mathrm{Ind}_{\langle\sigma\rangle}^D k$ and the latter is isomorphic to $kD(1+\sigma)$ by (3.3), Lemma 3.1 follows in case (i). Case (ii) is proved using the set X_τ of left cosets of $\langle\tau\rangle$ in D instead. □

4. Endo-trivial modules for D in characteristic 2

As before, let k be an algebraically closed field of characteristic 2. Since D is a 2-group, it follows from [12] that the kD-modules V with stable endomorphism ring $\underline{\mathrm{End}}_{kD}(V) \cong k$ are precisely the endo-trivial kD-modules, i.e. the kD-modules V whose endomorphism ring over k, $\mathrm{End}_k(V)$, is as kD-module stably isomorphic to the trivial kD-module k. The latter modules have been completely classified in [13] (see also [14]). We will use this description to determine the location of the

indecomposable endo-trivial kD-modules in the stable Auslander-Reiten quiver of kD.

Remark 4.1. Let $z = (\sigma\tau)^{2^{d-2}}$ be the involution in the center of D. The poset of all elementary abelian subgroups of D of rank at least 2 consists of two conjugacy classes of maximal elementary abelian subgroups of rank exactly 2. These conjugacy classes are represented by $K_1 = \langle \sigma, z\rangle$ and $K_2 = \langle \tau, z\rangle$. Let $T(D)$ denote the group of equivalence classes of endo-trivial kD-modules as in [13]. Consider the map

$$\Xi : T(D) \to \mathbb{Z} \times \mathbb{Z} \tag{4.1}$$

defined by $\Xi([M]) = (a_1, a_2)$ when $\mathrm{Res}^D_{K_i} M \cong \Omega^{a_i}_{K_i}(k) \oplus F_{M,i}$ for some free kK_i-module $F_{M,i}$ for $i = 1, 2$. In particular, $\Xi([k]) = (0, 0)$ and $\Xi([\Omega^m(M)]) = \Xi([M]) + (m, m)$ for all endo-trivial kD-modules M and all integers m. By [13, Thm. 5.4], Ξ is injective and the image of Ξ is generated by $(1, 1)$ and $(1, -1)$ (and also by $(1, 1)$ and $(-1, 1)$).

As in §3, let $E_\sigma = kD(1 + \sigma)/\mathrm{soc}(kD)$ and let $E_\tau = kD(1 + \tau)/\mathrm{soc}(kD)$. By (3.2), $\mathrm{rad}(kD)/\mathrm{soc}(kD) \cong E_\sigma \oplus E_\tau$. The almost split sequence ending in $\Omega^{-1}(k) = kD/\mathrm{soc}(kD)$ has thus the form

$$0 \to \Omega(k) \to kD \oplus E_\sigma \oplus E_\tau \xrightarrow{\mu_{-1}} \Omega^{-1}(k) \to 0 \tag{4.2}$$

where $\mu_{-1}|_{E_\nu}$ is the rightmost vertical homomorphism f_ν in the diagram (3.4) and $\mu_{-1}|_{kD}$ is the natural projection. It follows for example from [3, Lemma 5.4] that E_σ and E_τ are endo-trivial. Moreover, $\Xi([E_\sigma]) = (1, -1)$ and $\Xi([E_\tau]) = (-1, 1)$. In particular, $T(D)$ is generated by $[\Omega(k)]$ and $[E_\sigma]$ (and also by $[\Omega(k)]$ and $[E_\tau]$).

Let $\nu \in \{\sigma, \tau\}$, and define $A_{\nu,0} = k$ and $A_{\nu,1} = \Omega(E_\nu)$. For $n \geq 2$, define $A_{\nu,n}$ to be the unique indecomposable kD-module, up to isomorphism, in the equivalence class of the endo-trivial kD-module $A_{\nu,1} \otimes_k A_{\nu,n-1}$. Then the trivial simple kD-module $k = A_{\sigma,0} = A_{\tau,0}$ together with the kD-modules $A_{\sigma,n}$, $A_{\tau,n}$ for $n \geq 1$ give a complete set of representatives of the Ω-orbits of the indecomposable endo-trivial kD-modules. We have $\Xi([A_{\sigma,n}]) = (2n, 0)$ and $\Xi([A_{\tau,n}]) = (0, 2n)$ for all $n \geq 0$.

Lemma 4.2. *The finitely generated indecomposable endo-trivial kD-modules are exactly the modules in the two components of the stable Auslander-Reiten quiver of kD containing the trivial simple kD-module k and $\Omega(k)$.*

More precisely, let $A_{\sigma,n}$ and $A_{\tau,n}$ be as in Remark 4.1 for $n \geq 0$. Then the almost split sequence ending in k has the form

$$0 \to \Omega^2(k) \to A_{\sigma,1} \oplus A_{\tau,1} \xrightarrow{\mu_1} k \to 0. \tag{4.3}$$

Let $\nu \in \{\sigma, \tau\}$ and let $n \geq 1$. Then the almost split sequence ending in $A_{\nu,n}$ has the form

$$0 \to \Omega^2(A_{\nu,n}) \to A_{\nu,n+1} \oplus \Omega^2(A_{\nu,n-1}) \xrightarrow{\mu_{\nu,n+1}} A_{\nu,n} \to 0. \tag{4.4}$$

Proof. Since Ω defines an equivalence of the stable module category of finitely generated kD-modules with itself, we can apply Ω to the almost split sequence (4.2) to obtain the almost split sequence ending in k up to free direct summands of the middle term. Since the sequence (4.2) is the only almost split sequence having kD as a summand of the middle term, the almost split sequence ending in k is as in (4.3).

Given an indecomposable kD-module M of odd k-dimension, it follows from [3, Thm. 3.6 and Cor. 4.7] that

$$0 \to \Omega^2(k) \otimes_k M \to (A_{\sigma,1} \otimes_k M) \oplus (A_{\tau,1} \otimes_k M) \to M \to 0 \tag{4.5}$$

is the almost split sequence ending in M modulo projective direct summands. Since all endo-trivial kD-modules have odd k-dimension, we can apply the sequence (4.5) to $M = A_{\nu,n}$ for $\nu \in \{\sigma, \tau\}$ and all $n \geq 1$. This means that modulo free direct summands the almost split sequence ending in $A_{\nu,n}$ has the form

$$0 \to \Omega^2(k) \otimes_k A_{\nu,n} \to (A_{\sigma,1} \otimes_k A_{\nu,n}) \oplus (A_{\tau,1} \otimes_k A_{\nu,n}) \to A_{\nu,n} \to 0. \tag{4.6}$$

Note that $\Xi([A_{\nu',1} \otimes_k A_{\nu,n}]) = (2,2) + \Xi([A_{\nu,n-1}]) = \Xi([\Omega^2(A_{\nu,n-1})])$ if $\{\nu, \nu'\} = \{\sigma, \tau\}$. Since the sequence (4.2) is the only almost split sequence having kD as a summand of the middle term, it follows that the almost split sequence ending in $A_{\nu,n}$ is as in (4.4). This completes the proof of Lemma 4.2. □

Lemma 4.3. *Let $\nu \in \{\sigma, \tau\}$, let $n \geq 0$ and let $A_{\nu,n}$ be as in Remark 4.1. Then $\dim_k A_{\nu,n} = n2^{d-1} + 1$ and $\mathrm{Res}^D_C A_{\nu,n} \cong k \oplus (kC)^{n2^{d-2}}$ for $C \in \{\langle\sigma\rangle, \langle\tau\rangle\}$. Moreover, there is a short exact sequence of kD-modules*

$$0 \to \mathrm{Ind}^D_{\langle\nu\rangle} k \to A_{\nu,n+1} \to A_{\nu,n} \to 0. \tag{4.7}$$

Proof. When we restrict the almost split sequences (4.3) and (4.4) to the elementary abelian subgroups K_1 and K_2 from Remark 4.1, it follows that the resulting short exact sequences of kK_i-modules split for $i = 1, 2$. Define $\phi_{\nu,1} : A_{\nu,1} \to k$ to be the restriction of the homomorphism μ_1 in (4.3) to the component $A_{\nu,1}$, and for $n \geq 1$ define $\phi_{\nu,n+1} : A_{\nu,n+1} \to A_{\nu,n}$ to be the restriction of the homomorphism $\mu_{\nu,n+1}$ in (4.4) to the component $A_{\nu,n+1}$. For $n \geq 1$, let $\Phi_{\nu,n} : A_{\nu,n} \to k$ be the composition $\Phi_{\nu,n} = \phi_{\nu,1} \circ \phi_{\nu,2} \circ \ldots \circ \phi_{\nu,n}$. Then it follows that the homomorphism $\Phi_n : A_{\sigma,n} \oplus A_{\tau,n} \to k$, which restricted to $A_{\nu,n}$ is given by $\Phi_{\nu,n}$, splits when viewed as a homomorphism of kK_i-modules for $i = 1, 2$. Since $\Xi([A_{\sigma,n}]) = (2n, 0)$ and $\Xi([A_{\tau,n}]) = (0, 2n)$, this implies that for all $n \geq 1$ we have a short exact sequence of kD-modules of the form

$$0 \to \Omega^{2n}(k) \to A_{\sigma,n} \oplus A_{\tau,n} \xrightarrow{\Phi_n} k \to 0.$$

Inductively, we see that $\Omega^{2n}(k)$ has k-dimension $n2^d + 1$. Because the k-dimensions of $A_{\sigma,n}$ and $A_{\tau,n}$ coincide, this implies that $\dim_k A_{\nu,n} = \frac{1}{2}(n2^d + 2) = n2^{d-1} + 1$.

Let $C \in \{\langle\sigma\rangle, \langle\tau\rangle\}$. Since $E_\nu = kD(1+\nu)/\mathrm{soc}(kD)$, it follows that $\mathrm{Res}^D_C E_\nu$ is stably isomorphic to k. Hence we obtain for $A_{\nu,1} = \Omega(E_\nu)$ that $\mathrm{Res}^D_C A_{\nu,1}$ also is stably isomorphic to k. Therefore, it follows by induction from the almost split sequence (4.4) that $\mathrm{Res}^D_C A_{\nu,n+1}$ is stably isomorphic to k for all $n \geq 1$. Comparing k-dimensions it follows that $\mathrm{Res}^D_C A_{\nu,n} \cong k \oplus (kC)^{n2^{d-2}}$ for all $n \geq 0$.

To construct a short exact sequence of the form (4.7), recall that the almost split sequence (4.3) is obtained by applying Ω to the almost split sequence (4.2). Since the restriction of the homomorphism μ_{-1} in (4.2) to the component E_ν is the same as the homomorphism f_ν in the diagram (3.4), the restriction of the homomorphism μ_1 in (4.3) to the component $A_{\nu,1}$ is the same as $\Omega(f_\nu)$. By (3.5) we have $\mathrm{Ker}(\Omega(f_\nu)) \cong$

$kD(1+\nu)$, which implies by (3.3) that there is a short exact sequence of kD-modules of the form

$$0 \to \mathrm{Ind}_{\langle\nu\rangle}^D\, k \to A_{\nu,1} \to k \to 0. \tag{4.8}$$

Now let $n \geq 1$. Tensoring the sequence (4.8) with $A_{\nu,n}$, we obtain a short exact sequence of kD-modules of the form

$$0 \to (\mathrm{Ind}_{\langle\nu\rangle}^D\, k) \otimes_k A_{\nu,n} \to A_{\nu,1} \otimes_k A_{\nu,n} \to A_{\nu,n} \to 0. \tag{4.9}$$

Since $(\mathrm{Ind}_{\langle\nu\rangle}^D\, k) \otimes_k A_{\nu,n} \cong \mathrm{Ind}_{\langle\nu\rangle}^D\, (\mathrm{Res}_{\langle\nu\rangle}^D\, A_{\nu,n})$ and $\mathrm{Res}_{\langle\nu\rangle}^D\, A_{\nu,n} \cong k \oplus (k\langle\nu\rangle)^{n2^{d-2}}$, it follows that

$$(\mathrm{Ind}_{\langle\nu\rangle}^D\, k) \otimes_k A_{\nu,n} \cong \mathrm{Ind}_{\langle\nu\rangle}^D\, k \;\oplus (kD)^{n2^{d-2}}.$$

By definition, $A_{\nu,n+1}$ is the unique indecomposable kD-module, up to isomorphism, in the equivalence class of the endo-trivial kD-module $A_{\nu,1} \otimes_k A_{\nu,n}$. Comparing k-dimensions it follows that $\dim_k A_{\nu,1} \otimes_k A_{\nu,n} = n2^{d-2} \cdot 2^d + \dim_k A_{\nu,n+1}$. Hence

$$A_{\nu,1} \otimes_k A_{\nu,n} \cong A_{\nu,n+1} \oplus (kD)^{n2^{d-2}}.$$

By splitting off the free kD-module $(kD)^{n2^{d-2}}$ from the first and second term of the short exact sequence (4.9) we obtain a short exact sequence of kD-modules of the form (4.7). □

5. Universal deformation rings for D and nilpotent blocks

In this section, we prove Theorem 1.2 and Corollary 1.3. We use the notations of the previous sections. In particular, k is an algebraically closed field of characteristic 2, W is the ring of infinite Witt vectors over k and F is the fraction field of W.

For $\nu \in \{\sigma, \tau\}$ and $n \geq 0$, let $A_{\nu,n}$ be as in Remark 4.1. We first analyze all extensions of $A_{\nu,n}$ by $A_{\nu,n}$ by using the extensions of k by k which are described in the following remark.

Remark 5.1. Suppose N is a kD-module which lies in a non-split short exact sequence of kD-modules

$$0 \to k \to N \to k \to 0. \tag{5.1}$$

Then N is isomorphic to N_λ for some $\lambda \in k^* \cup \{\sigma, \tau\}$ where a representation φ_λ of N_λ is given by the following 2×2 matrices over k.

a. If $\lambda \in k^*$ then $\varphi_\lambda(\sigma) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\varphi_\lambda(\tau) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$.

b. If $\{\lambda, \lambda'\} = \{\sigma, \tau\}$ then $\varphi_\lambda(\lambda) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\varphi_\lambda(\lambda') = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

We obtain the following restrictions of N_λ to the subgroups $\langle\sigma\rangle$ and $\langle\tau\rangle$ of D.

- In case (a), $\mathrm{Res}_C^D\, N_\lambda \cong kC$ for $C \in \{\langle\sigma\rangle, \langle\tau\rangle\}$.
- In case (b), $\mathrm{Res}_{\langle\lambda\rangle}^D\, N_\lambda \cong k\langle\lambda\rangle$ and $\mathrm{Res}_{\langle\lambda'\rangle}^D\, N_\lambda \cong k^2$.

Lemma 5.2. *Let $\nu \in \{\sigma, \tau\}$, let $n \geq 0$, and let $A_{\nu,n}$ be as in Remark 4.1. Then $\mathrm{Ext}^1_{kD}(A_{\nu,n}, A_{\nu,n}) \cong k^2$. Suppose $Z_{\nu,n}$ is a kD-module which lies in a non-split short exact sequence of kD-modules*

$$0 \to A_{\nu,n} \to Z_{\nu,n} \to A_{\nu,n} \to 0. \tag{5.2}$$

Then $Z_{\nu,n}$ is isomorphic to $B_{\nu,n,\lambda} = A_{\nu,n} \otimes_k N_\lambda$ for some $\lambda \in k^* \cup \{\sigma, \tau\}$ where N_λ is as in Remark 5.1. Let $\{\nu, \nu'\} = \{\sigma, \tau\}$.

i. *If $\lambda \in k^* \cup \{\nu\}$ then $B_{\nu,n,\lambda} \cong N_\lambda \oplus (kD)^n$.*
ii. *If $\lambda = \nu'$ then $B_{\nu,0,\nu'} \cong N_{\nu'}$ and for $n \geq 1$, $B_{\nu,n,\nu'}$ is a non-split extension of $B_{\nu,n-1,\nu'}$ by $kD(1+\nu) \oplus kD(1+\nu)$. Moreover, $\mathrm{Res}^D_{\langle\nu'\rangle} B_{\nu,n,\nu'} \cong (k\langle\nu'\rangle)^{n2^{d-1}+1}$ and $\mathrm{Res}^D_{\langle\nu\rangle} B_{\nu,n,\nu'} \cong k^2 \oplus (k\langle\nu\rangle)^{n2^{d-1}}$.*

Proof. Since $A_{\nu,n}$ is endo-trivial, it follows from [3, Thm. 2.6] that the natural homomorphism

$$\mathrm{Ext}^1_{kD}(k,k) \to \mathrm{Ext}^1_{kD}(A_{\nu,n}, A_{\nu,n}) \tag{5.3}$$

resulting from tensoring short exact sequences of the form (5.1) with $A_{\nu,n}$ is a monomorphism. Moreover,

$$\mathrm{Ext}^1_{kD}(A_{\nu,n}, A_{\nu,n}) \cong \mathrm{H}^1(D, A^*_{\nu,n} \otimes_k A_{\nu,n}) \cong \mathrm{H}^1(D,k) \cong \mathrm{Ext}^1_{kD}(k,k) \cong k^2$$

where the second isomorphism follows since $A_{\nu,n}$ is endo-trivial. Hence the homomorphism in (5.3) is an isomorphism, which means that we only need to prove the descriptions of $B_{\nu,n,\lambda} = A_{\nu,n} \otimes_k N_\lambda$ given in (i) and (ii) of the statement of Lemma 5.2. For $n = 0$ this follows from Remark 5.1 since $A_{\nu,0} = k$ and so $B_{\nu,0,\lambda} = N_\lambda$.

By Lemma 4.3, there is a short exact sequence of kD-modules of the form (4.7). Tensoring this sequence with N_λ over k gives a short exact sequence of kD-modules of the form

$$0 \to (\mathrm{Ind}^D_{\langle\nu\rangle} k) \otimes_k N_\lambda \to B_{\nu,n+1,\lambda} \to B_{\nu,n,\lambda} \to 0 \tag{5.4}$$

where $(\mathrm{Ind}^D_{\langle\nu\rangle} k) \otimes_k N_\lambda \cong \mathrm{Ind}^D_{\langle\nu\rangle} (\mathrm{Res}^D_{\langle\nu\rangle} N_\lambda)$.

Let first $\lambda \in k^* \cup \{\nu\}$. Then $\mathrm{Res}^D_{\langle\nu\rangle} N_\lambda \cong k\langle\nu\rangle$ and hence $\mathrm{Ind}^D_{\langle\nu\rangle} (\mathrm{Res}^D_{\langle\nu\rangle} N_\lambda) \cong kD$. It follows by induction that $B_{\nu,n+1,\lambda} \cong N_\lambda \oplus (kD)^{n+1}$, which proves (i).

Now let $\lambda = \nu'$. Then $\mathrm{Res}^D_{\langle\nu\rangle} N_{\nu'} \cong k^2$ and hence

$$\mathrm{Ind}^D_{\langle\nu\rangle} (\mathrm{Res}^D_{\langle\nu\rangle} N'_\nu) \cong \mathrm{Ind}^D_{\langle\nu\rangle} k^2 \cong kD(1+\nu) \oplus kD(1+\nu).$$

Thus $B_{\nu,n+1,\nu'}$ is an extension of $B_{\nu,n,\nu'}$ by $kD(1+\nu) \oplus kD(1+\nu)$. Since $\mathrm{Res}^D_{\langle\nu'\rangle} kD(1+\nu) \cong (k\langle\nu'\rangle)^{2^{d-2}}$, it follows by induction that $\mathrm{Res}^D_{\langle\nu'\rangle} B_{\nu,n+1,\nu'} \cong (k\langle\nu'\rangle)^{(n+1)2^{d-1}+1}$. On the other hand, $\mathrm{Res}^D_{\langle\nu\rangle} kD(1+\nu) \cong k^2 \oplus (k\langle\nu\rangle)^{2^{d-2}-1}$. Hence the restriction of the left term in the short exact sequence (5.4) to $\langle\nu\rangle$ is isomorphic to $k^4 \oplus (k\langle\nu\rangle)^{2^{d-1}-2}$. Thus by induction, $\mathrm{Res}^D_{\langle\nu\rangle} B_{\nu,n+1,\nu'}$ lies in a short exact sequence of $k\langle\nu\rangle$-modules of the form

$$0 \to k^4 \oplus (k\langle\nu\rangle)^{2^{d-1}-2} \to \mathrm{Res}^D_{\langle\nu\rangle} B_{\nu,n+1,\nu'} \to k^2 \oplus (k\langle\nu\rangle)^{n2^{d-1}} \to 0.$$

Since $\mathrm{Res}^D_{\langle\nu\rangle} B_{\nu,n+1,\nu'}$ is an extension of $\mathrm{Res}^D_{\langle\nu\rangle} A_{\nu,n+1}$ by itself and since $\mathrm{Res}^D_{\langle\nu\rangle} A_{\nu,n+1} \cong k\oplus(k\langle\nu\rangle)^{(n+1)2^{d-2}}$ by Lemma 4.3, it follows that $\mathrm{Res}^D_{\langle\nu\rangle} B_{\nu,n+1,\nu'} \cong k^2\oplus(k\langle\nu\rangle)^{(n+1)2^{d-1}}$. In particular, the sequence (5.4) does not split when $\lambda = \nu'$, since it does not split when restricted to $\langle\nu\rangle$. This proves (ii). $\square$

The next result uses the restrictions of $A_{\nu,n}$ and $B_{\nu,n,\lambda}$ to the cyclic subgroups $\langle\sigma\rangle$ and $\langle\tau\rangle$ of D.

Lemma 5.3. *Let $\nu \in \{\sigma, \tau\}$, let $n \geq 0$, and let $A_{\nu,n}$ be as in Remark 4.1. Then there is a surjective W-algebra homomorphism $\alpha : W[\mathbb{Z}/2 \times \mathbb{Z}/2] \to R(D, A_{\nu,n})$ in $\mathcal{C}$.*

Proof. Let $R = R(D, A_{\nu,n})$ and let $U_{\nu,n}$ be a universal lift of $A_{\nu,n}$ over R. Let $C_\sigma = \langle\sigma\rangle$ and $C_\tau = \langle\tau\rangle$, and let $C \in \{C_\sigma, C_\tau\}$. By Lemma 4.3, $\mathrm{Res}^D_C A_{\nu,n} \cong k \oplus (kC)^{n2^{d-2}}$. In particular, $\mathrm{Res}^D_C A_{\nu,n}$ is a kC-module with stable endomorphism ring k, and hence it has a universal deformation ring. Moreover, $R(C, \mathrm{Res}^D_C A_{\nu,n}) \cong W[\mathbb{Z}/2]$. Let $U_{\nu,n,C}$ be a universal lift of $\mathrm{Res}^D_C A_{\nu,n}$ over $W[\mathbb{Z}/2]$. Then there exists a unique W-algebra homomorphism $\alpha_C : W[\mathbb{Z}/2] \to R$ in $\mathcal{C}$ such that $\mathrm{Res}^D_C U_{\nu,n} \cong R \otimes_{W[\mathbb{Z}/2],\alpha_C} U_{\nu,n,C}$. Since the completed tensor product over W is the coproduct in the category $\mathcal{C}$, we get a W-algebra homomorphism

$$\alpha = \alpha_{C_\sigma} \otimes \alpha_{C_\tau} : W[\mathbb{Z}/2] \otimes_W W[\mathbb{Z}/2] \to R$$

in $\mathcal{C}$ with $\alpha(x_1 \otimes x_2) = \alpha_{C_\sigma}(x_1)\, \alpha_{C_\tau}(x_2)$.

By Lemma 5.2, every non-trivial lift of $A_{\nu,n}$ over the dual numbers $k[\epsilon]/(\epsilon^2)$ is as kD-module isomorphic to $B_{\nu,n,\lambda}$ for some $\lambda \in k^* \cup \{\sigma, \tau\}$. The description of these modules shows that their restrictions to C_σ and C_τ are as follows:

i. If $\lambda \in k^*$ then $\mathrm{Res}^D_C B_{\nu,n,\lambda} = (kC)^{n2^{d-1}+1}$ for $C \in \{C_\sigma, C_\tau\}$.
ii. If $\{\lambda, \lambda'\} = \{\sigma, \tau\}$, then $\mathrm{Res}^D_{C_\lambda} B_{\nu,n,\lambda} = (kC_\lambda)^{n2^{d-1}+1}$ and $\mathrm{Res}^D_{C_{\lambda'}} B_{\nu,n,\lambda} = k^2 \oplus (kC_{\lambda'})^{n2^{d-1}}$.

Note that for $C \in \{C_\sigma, C_\tau\}$, $k^2 \oplus (kC)^{n2^{d-1}}$ corresponds to the trivial lift of $\mathrm{Res}^D_C A_{\nu,n}$ over $k[\epsilon]/(\epsilon^2)$, and $(kC)^{n2^{d-1}+1}$ corresponds to a non-trivial lift of $\mathrm{Res}^D_C A_{\nu,n}$ over $k[\epsilon]/(\epsilon^2)$.

Let $\lambda \in k^* \cup \{\sigma, \tau\}$, and let $f_\lambda : R \to k[\epsilon]/(\epsilon^2)$ be a morphism corresponding to a non-trivial lift of $A_{\nu,n}$ over $k[\epsilon]/(\epsilon^2)$ with underlying kD-module structure given by $B_{\nu,n,\lambda}$. Then $f_\lambda \circ \alpha = f_\lambda \circ (\alpha_{C_\sigma} \otimes \alpha_{C_\tau})$ corresponds to the pair of lifts of the pair $(\mathrm{Res}^D_{C_\sigma} A_{\nu,n}, \mathrm{Res}^D_{C_\tau} A_{\nu,n})$ over $k[\epsilon]/(\epsilon^2)$ with underlying kD-module structure given by the pair $(\mathrm{Res}^D_{C_\sigma} B_{\nu,n,\lambda}, \mathrm{Res}^D_{C_\tau} B_{\nu,n,\lambda})$. Hence (i) and (ii) above imply that if f runs through the morphisms $R \to k[\epsilon]/(\epsilon^2)$, then $f \circ \alpha$ runs through the morphisms $W[\mathbb{Z}/2] \otimes_W W[\mathbb{Z}/2] \to k[\epsilon]/(\epsilon^2)$. This implies α is surjective. $\square$

We next determine how many non-isomorphic lifts $A_{\nu,n}$ has over W.

Lemma 5.4. *Let $\nu \in \{\sigma, \tau\}$, let $n \geq 0$, and let $A_{\nu,n}$ be as in Remark 4.1. Then $A_{\nu,n}$ has four pairwise non-isomorphic lifts over W corresponding to four distinct morphisms $R(D, A_{\nu,n}) \to W$ in $\mathcal{C}$.*

Proof. We use Lemmas 2.3 and 3.1 to prove this. If $n=0$ then $A_{\nu,0}$ is the trivial simple kD-module k which has four pairwise non-isomorphic lifts over W whose F-characters are given by the four ordinary irreducible characters of degree one $\chi_1,\chi_2,\chi_3,\chi_4$ (see Table 1). By Lemma 4.3, there is a short exact sequence of kD-modules of the form

$$0\to \mathrm{Ind}_{\langle\nu\rangle}^D k\to A_{\nu,n+1}\to A_{\nu,n}\to 0.$$

By Frobenius reciprocity and the Eckman-Shapiro Lemma, we have for all $n\geq 0$

$$\begin{array}{rcl}\mathrm{Hom}_{kD}(A_{\nu,n},\mathrm{Ind}_{\langle\nu\rangle}^D k) & \cong & \mathrm{Hom}_{k\langle\nu\rangle}(\mathrm{Res}_{\langle\nu\rangle}^D A_{\nu,n},k)\;\cong\; k^{n2^{d-2}+1}\quad\text{and}\\ \mathrm{Ext}^1_{kD}(A_{\nu,n},\mathrm{Ind}_{\langle\nu\rangle}^D k) & \cong & \mathrm{Ext}^1_{k\langle\nu\rangle}(\mathrm{Res}_{\langle\nu\rangle}^D A_{\nu,n},k)\;\cong\; k\end{array}$$

where the second isomorphisms follow since $\mathrm{Res}_{\langle\nu\rangle}^D A_{\nu,n}\cong k\oplus(k\langle\nu\rangle)^{n2^{d-2}}$ by Lemma 4.3.

Let $\rho=\sum_{\ell=0}^{d-3}\rho_\ell$ for ρ_ℓ as in (3.6). Then every FD-module T with F-character ρ satisfies by (3.7)

$$\dim_F \mathrm{End}_{FD}(T)=\sum_{\ell=0}^{d-3}\dim_F\mathrm{End}_{FD}(V_\ell)=\sum_{\ell=0}^{d-3}2^\ell=2^{d-2}-1.$$

Let $(c,d)=(3,4)$ if $\nu=\sigma$, and let $(c,d)=(4,3)$ if $\nu=\tau$. Assume by induction that $A_{\nu,n}$ has four pairwise non-isomorphic lifts over W whose F-characters are given by

i. $\chi_1+m(\chi_1+\chi_c)+m(\chi_2+\chi_d)+2m\,\rho$, or
$\chi_c+m(\chi_1+\chi_c)+m(\chi_2+\chi_d)+2m\,\rho$, or
ii. $\chi_2+m(\chi_1+\chi_c)+m(\chi_2+\chi_d)+2m\,\rho$, or
$\chi_d+m(\chi_1+\chi_c)+m(\chi_2+\chi_d)+2m\,\rho$

if $n=2m$ for some $m\geq 0$, and by

i′. $\chi_1+m(\chi_1+\chi_c)+(m+1)(\chi_2+\chi_d)+(2m+1)\rho$, or
$\chi_c+m(\chi_1+\chi_c)+(m+1)(\chi_2+\chi_d)+(2m+1)\rho$, or
ii′. $\chi_2+(m+1)(\chi_1+\chi_c)+m(\chi_2+\chi_d)+(2m+1)\rho$, or
$\chi_d+(m+1)(\chi_1+\chi_c)+m(\chi_2+\chi_d)+(2m+1)\rho$

if $n=2m+1$ for some $m\geq 0$. By Lemma 3.1, $\mathrm{Ind}_{\langle\nu\rangle}^D k$ has two pairwise non-isomorphic lifts over W with F-characters $\chi_1+\chi_c+\rho$ or $\chi_2+\chi_d+\rho$. If $\mathcal{A}_{\nu,n}$ is a lift of $A_{\nu,n}$ over W with F-character as in (i) or (ii′), let $\mathcal{I}$ be a lift of $\mathrm{Ind}_{\langle\nu\rangle}^D k$ over W with F-character $\chi_2+\chi_d+\rho$. If $\mathcal{A}_{\nu,n}$ is a lift of $A_{\nu,n}$ over W with F-character as in (ii) or (i′), let $\mathcal{I}$ be a lift of $\mathrm{Ind}_{\langle\nu\rangle}^D k$ over W with F-character $\chi_1+\chi_c+\rho$. Then we obtain for $n=2m$,

$$\dim_F\mathrm{Hom}_{FD}(F\otimes_W\mathcal{A}_{\nu,n},F\otimes_W\mathcal{I})=2m\,2^{d-2}=n2^{d-2},$$

and for $n=2m+1$,

$$\dim_F\mathrm{Hom}_{FD}(F\otimes_W\mathcal{A}_{\nu,n},F\otimes_W\mathcal{I})=(2m+1)\,2^{d-2}=n2^{d-2}.$$

Hence by Lemma 2.3, $A_{\nu,n+1}$ has four pairwise non-isomorphic lifts over W whose F-characters are as in (i),(ii), respectively (i′), (ii′), if we replace n by $n+1$. □

Proof of Theorem 1.2. Part (i) follows from Lemma 4.2. For part (ii), let V be an arbitrary finitely generated endo-trivial kD-module. By Lemma 2.2(ii), it is enough to determine $R(D,V)$ in case V is indecomposable. This means by Remark 4.1 that V is in the Ω-orbit of k or of $A_{\sigma,n}$ or of $A_{\tau,n}$ for some $n \geq 1$.

Let $\nu \in \{\sigma, \tau\}$ and let $n \geq 0$. It follows by Lemmas 5.3 and 5.4 that there is a surjective morphism

$$\alpha : W[\mathbb{Z}/2 \times \mathbb{Z}/2] \to R(D, A_{\nu,n})$$

in $\mathcal{C}$ and that there are four distinct morphisms $R(D, A_{\nu,n}) \to W$ in $\mathcal{C}$. Hence $\mathrm{Spec}(R(D, A_{\nu,n}))$ contains all four points of the generic fiber of $\mathrm{Spec}(W[\mathbb{Z}/2 \times \mathbb{Z}/2])$. Since the Zariski closure of these four points is all of $\mathrm{Spec}(W[\mathbb{Z}/2\times\mathbb{Z}/2])$, this implies that $R(D, A_{\nu,n})$ must be isomorphic to $W[\mathbb{Z}/2\times\mathbb{Z}/2]$. By Lemma 2.2, it follows that $R(D,V) \cong W[\mathbb{Z}/2 \times \mathbb{Z}/2]$ for every finitely generated endo-trivial kD-module V.

Let U be a universal lift of V over $R = R(D,V)$. Since V is endo-trivial, the rank of U as a free R-module is odd. This implies that as RD-modules

$$U^* \otimes_R U \cong R \oplus L$$

where D acts trivially on R and L is some RD-module which is free over R. Since $U^* \otimes_R U$ is a lift of $V^* \otimes_k V$ over R and V is endo-trivial, this implies that $k \otimes_R L$ is isomorphic to a free kD-module. Hence L is a free RD-module, which implies that U is endo-trivial. □

We now turn to nilpotent blocks and the proof of Corollary 1.3.

Remark 5.5. Keeping the previous notation, let G be a finite group and let B be a nilpotent block of kG with defect group D. By [9], this means that whenever (Q,f) is a B-Brauer pair then the quotient $N_G(Q,f)/C_G(Q)$ is a 2-group. In other words, for all subgroups Q of D and for all block idempotents f of $kC_G(Q)$ associated with B, the quotient of the stabilizer $N_G(Q,f)$ of f in $N_G(Q)$ by the centralizer $C_G(Q)$ is a 2-group. In [25], Puig rephrased this definition using the theory of local pointed groups.

The main result of [25] implies that the nilpotent block B is Morita equivalent to kD. In [26, Thm. 8.2], Puig showed that the converse is also true in a very strong way. Namely, if B' is another block such that there is a stable equivalence of Morita type between B and B', then B' is also nilpotent. Hence Corollary 1.3 can be applied in particular if there is only known to be a stable equivalence of Morita type between B and kD.

Proof of Corollary 1.3. Let $\hat{B}$ be the block of WG corresponding to B. Then $\hat{B}$ is also nilpotent, and by [25, §1.4], $\hat{B}$ is Morita equivalent to WD. Suppose V is a finitely generated B-module, and V' is the kD-module corresponding to V under this Morita equivalence. Then V has stable endomorphism ring k if and only if V' has stable endomorphism ring k. Moreover, it follows for example from [4, Prop. 2.5] that $R(G,V) \cong R(D,V')$. By Theorem 1.2, this implies that $R(G,V) \cong W[\mathbb{Z}/2 \times \mathbb{Z}/2]$. □

References

[1] J. L. Alperin, Local representation theory. Modular representations as an introduction to the local representation theory of finite groups. Cambridge Studies in Advanced Mathematics, vol. 11, Cambridge University Press, Cambridge, 1986.

[2] J. L. Alperin, Lifting endo-trivial modules. J. Group Theory 4 (2001), 1–2.

[3] M. Auslander and J. F. Carlson, Almost-split sequences and group rings. J. Algebra 103 (1986), 122-140.

[4] F. M. Bleher, Universal deformation rings and Klein four defect groups. Trans. Amer. Math. Soc. 354 (2002), 3893–3906.

[5] F. M. Bleher, Universal deformation rings and dihedral defect groups. In press, Trans. Amer. Math. Soc.

[6] F. M. Bleher and T. Chinburg, Universal deformation rings and cyclic blocks. Math. Ann. 318 (2000), 805–836.

[7] F. M. Bleher and T. Chinburg, Universal deformation rings need not be complete intersections. C. R. Math. Acad. Sci. Paris 342 (2006), 229–232.

[8] F. M. Bleher and T. Chinburg, Universal deformation rings need not be complete intersections, Math. Ann. 337 (2007), 739–767.

[9] M. Broué and L. Puig, A Frobenius theorem for blocks. Invent. Math. 56 (1980), 117–128.

[10] M. C. R. Butler and C. M. Ringel, Auslander-Reiten sequences with few middle terms and applications to string algebras. Comm. Algebra 15 (1987), 145–179.

[11] J. Byszewski, A universal deformation ring which is not a complete intersection ring. C. R. Math. Acad. Sci. Paris 343 (2006), 565–568.

[12] J. F. Carlson, A characterization of endotrivial modules over p-groups. Manuscripta Math. 97 (1998), 303–307.

[13] J. F. Carlson and J. Thévenaz, Torsion endo-trivial modules. Algebr. Represent. Theory 3 (2000), 303–335.

[14] J. F. Carlson and J. Thévenaz, The classification of endo-trivial modules. Invent. Math. 158 (2004), 389–411.

[15] J. F. Carlson and J. Thévenaz, The classification of torsion endo-trivial modules. Ann. of Math. (2) 162 (2005), 823–883.

[16] T. Chinburg, Can deformation rings of group representations not be local complete intersections? In: Problems from the Workshop on Automorphisms of Curves. Edited by Gunther Cornelissen and Frans Oort, with contributions by I. Bouw, T. Chinburg, G. Cornelissen, C. Gasbarri, D. Glass, C. Lehr, M. Matignon, F. Oort, R. Pries and S. Wewers. Rend. Sem. Mat. Univ. Padova 113 (2005), 129–177.

[17] E. Dade, Endo-permutation modules over p-groups. I; II. Ann. of Math. 107 (1978), 459–494; 108 (1978), 317–346.

[18] B. de Smit and H. W. Lenstra, Explicit construction of universal deformation rings. In: Modular Forms and Fermat's Last Theorem (Boston, MA, 1995), Springer-Verlag, Berlin-Heidelberg-New York, 1997, pp. 313–326.

[19] K. Erdmann, Blocks of Tame Representation Type and Related Algebras, Lecture Notes in Mathematics, vol. 1428, Springer-Verlag, Berlin-Heidelberg-New York, 1990.

[20] H. Krause, Maps between tree and band modules. J. Algebra 137 (1991), 186–194.

[21] M. Linckelmann, A derived equivalence for blocks with dihedral defect groups. J. Algebra 164 (1994), 244–255.

[22] M. Linckelmann, The source algebras of blocks with a Klein four defect group. J. Algebra 167 (1994), no. 3, 821–854.

[23] M. Linckelmann, On stable equivalences of Morita type. Derived equivalences for group rings, 221–232, Lecture Notes in Math., 1685, Springer, Berlin, 1998.

[24] B. Mazur, Deforming Galois representations. In: Galois groups over $\mathbb{Q}$ (Berkeley, CA, 1987), Springer-Verlag, Berlin-Heidelberg-New York, 1989, pp. 385–437.

[25] L. Puig, Nilpotent blocks and their source algebras. Invent. Math. 93 (1988), 77–116.
[26] L. Puig, On the local structure of Morita and Rickard equivalences between Brauer blocks. Progress in Mathematics, 178. Birkhäuser Verlag, Basel, 1999.
[27] J. Thévenaz, G-algebras and modular representation theory. Oxford University Press, Oxford, 1995.

Department of Mathematics, University of Iowa, Iowa City, IA 52242-1419
E-mail address: `fbleher@math.uiowa.edu`